TRINITY COLLEGE

Semester One Examination, 2021

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

A small body is initially at the origin. It is moving along the x-axis with velocity at time t seconds given by

$$v(t) = (t-2)^3 \quad cm/s$$

(a) Determine $x(t)$, a function for the displacement of the body at time t. (3 marks)

Solution
$x(t) = \int (t-2)^3 \, dt$ $x(t) = \dfrac{(t-2)^4}{4} + c$ $t = 0,\ x = 0 \quad \Rightarrow c = 4$ $x(t) = \dfrac{(t-2)^4}{4} + 4$
Specific behaviours
✓ reasonable attempt at using chain rule   ✓ correct antiderivative   ✓ correct displacement function

The small body is stationary when $t = T$.

(b) Determine the displacement of the body at $T + 3$ seconds. (2 marks)

Solution
$(T-2)^3 = 0 \quad \Rightarrow T = 2$ $T + 3 = 5$ $x(5) = \dfrac{(5-2)^4}{4} + 4 = \dfrac{97}{4}$
Specific behaviours
✓ correct value of T ✓ correct displacement

Question 2 **(10 marks)**

(a) Determine $\frac{d}{dx}\sin(\cos x)$. (2 marks)

Solution
$\cos(\cos x)\times(-\sin x)$
Specific behaviours
✓ indicates use of chain rule ✓ correct derivative

(b) Evaluate $f'\left(\frac{\pi}{4}\right)$ when $f(x)=\frac{x+\cos x}{\sin 2x}$. (4 marks)

Solution
$f'(x)=\frac{(1-\sin x)(\sin 2x)-(x+\cos x)(2\cos 2x)}{\sin^2 2x}$ $f'\left(\frac{\pi}{4}\right)=\frac{\left(1-\frac{\sqrt{2}}{2}\right)(1)-0}{1}$ $=1-\frac{\sqrt{2}}{2}$
Specific behaviours
✓ indicates use of quotient rule ✓ correct u' and v' ✓ correct derivative ✓ substitutes and simplifies

(c) Determine $f'(x)$ if $f(x)=\tan x$ and using $\tan x=\frac{\sin x}{\cos x}$ (4 marks)

Solution
$f(x)=\frac{\sin x}{\cos x}\Rightarrow f'(x)=\frac{\cos x\cos x-\sin x(-\sin x)}{(\cos x)^2}$ $\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}$
Specific behaviours
✓ recognises quotient rule required ✓ correct derivative ✓ simplifies numerator ✓ correct answer

Question 3 **(8 marks)**

(a) State three key characteristics of a chance experiment that make it suitable for modelling by a binomial random variable. (3 marks)

Solution
1. There are a fixed number of identical and independent trials. 2. There are only two possible outcomes for each trial ('success' and 'failure'). 3. The probability of 'success' is the same in every trial.
Specific behaviours
✓ identifies one characteristic ✓ identifies second characteristic ✓ identifies third characteristic

Research has shown that 30% of dogs aged over 13 years have some form of heart disease. A random sample of 40 dogs is selected from a large number of dogs of this age. Let X be the number of dogs in the sample with some form of heart disease.

(b) Explain why randomly selecting one dog and recording whether it has some form of heart disease is a Bernoulli trial. (1 mark)

Solution
It is a **chance experiment** (dog is selected at random) with **two possible outcomes** (dog has some form of heart disease, or it does not).
Specific behaviours
✓ mentions both bolded phrases, or their equivalent

(c) Write a numerical expression for the probability that 11 dogs in the sample have some form of heart disease. (2 marks)

Solution
$X \sim B(40, 0.3)$ $P(X = 11) = \binom{40}{11}(0.3)^{11}(0.7)^{29}$
Specific behaviours
✓ indicates binomial distribution ✓ correct expression

(d) State the mean and variance of X. (2 marks)

Solution
$E(X) = 40 \times 0.3 = 12$ $\text{Var}(X) = 12 \times 0.7 = 8.4$
Specific behaviours
✓ correct mean ✓ correct variance

Question 4 **(6 marks)**

Determine the area of the finite region bounded by $y = \sqrt{2x}$ and $y = \frac{x}{2}$.

Solution
Points of intersection: $$\sqrt{2x} = \frac{x}{2}$$ $$x^2 - 8x = 0$$ $$x = 0, \qquad x = 8$$ Area: $$A = \int_0^8 \sqrt{2x} - \frac{x}{2}\,dx$$ $$= \left[\frac{(2x)^{\frac{3}{2}}}{3} - \frac{x^2}{4}\right]_0^8$$ $$= \left[\frac{(16)^{\frac{3}{2}}}{3} - \frac{8\times 8}{4}\right] - 0$$ $$= \frac{64}{3} - 16$$ $$= \frac{16}{3}\ \mathrm{u}^2$$
Specific behaviours
✓ equates curves and squares ✓ points of intersection ✓ writes integral for area ✓ correct antiderivative ✓ substitutes ✓ simplifies to obtain area

Question 5 **(7 marks)**

A four-sided die has faces marked with the numbers $1, 2, 2$ and 3. All faces have an equal chance of landing face down after the die is rolled. A game, that costs $\$2$ to play, involves throwing the die twice and adding the two numbers that land face down. If the total score is 5, the player wins $\$7$, and otherwise they win nothing.

Let X be the total score obtained in one play of the game.

(a) Construct a probability distribution table for X. (3 marks)

Solution

x	2	3	4	5	6
$P(X = x)$	$^1/_{16}$	$^4/_{16}$	$^6/_{16}$	$^4/_{16}$	$^1/_{16}$

Specific behaviours

✓ table with label x and correct x values
✓ label $P(X = x)$ and at least two correct probabilities
✓ wholly correct pd table

(b) Determine $E(X)$. (1 mark)

Solution

$$E(X) = \frac{2 + 12 + 24 + 20 + 6}{16} = 4$$

Specific behaviours

✓ correct $E(X)$

Let Y be the net monetary loss, in dollars, of a player in **two** plays of the game.

(c) Determine $E(Y)$. (3 marks)

Solution

t	2	-5
$P(T = t)$	$^3/_4$	$^1/_4$

Let T be monetary loss in one game, then $E(T) = \frac{6-5}{4} = 0.25$.

Hence $E(Y) = 2 \times E(T) = \$0.50$.

Specific behaviours

✓ indicates possible losses with probabilities in one game
✓ indicates expected loss in one game
✓ calculates $E(Y)$

Question 6 **(5 marks)**

(a) Determine $\frac{d}{dx}(6x\sqrt{e^x})$. (2 marks)

Solution
$\frac{d}{dx}\left(6x \cdot e^{\frac{x}{2}}\right) = 6e^{\frac{x}{2}} + 3xe^{\frac{x}{2}}$
Specific behaviours
✓ uses product rule ✓ obtains correct result

(b) Hence, or otherwise, determine $\int \left(6x\sqrt{e^x}\right) dx$. (3 marks)

Solution
$\int \frac{d}{dx}\left(6xe^{\frac{x}{2}}\right) dx = \int 6e^{\frac{x}{2}}\, dx + \int 3xe^{\frac{x}{2}}\, dx$ $6xe^{\frac{x}{2}} = 12e^{\frac{x}{2}} + \int 3xe^{\frac{x}{2}}\, dx$ $2\int 3xe^{\frac{x}{2}}\, dx = \int \left(6x\sqrt{e^x}\right) dx = 12xe^{\frac{x}{2}} - 24e^{\frac{x}{2}} + c$
Specific behaviours
✓ integrates all terms of result from (a) ✓ uses fundamental theorem to simplify LHS ✓ obtains required result, with constant

Question 7 **(7 marks)**

The following table shows the probability distribution for the random variable T.

t	0	1
$P(T=t)$	$\frac{k}{4}+\frac{1}{10}$	$\frac{13}{10}-\frac{1}{5k}$

(a) Determine the value of the positive constant k and hence state $P(T=1)$. (4 marks)

Solution
$\frac{k}{4}+\frac{1}{10}+\frac{13}{10}-\frac{1}{5k}=1$ $5k^2+2k+26k-4=20k$ $5k^2+8k-4=0$ $(5k-2)(k+2)=0$ $k=\frac{2}{5}$ Hence $P(T=1)=\frac{13}{10}-\frac{1}{2}=\frac{4}{5}$
Specific behaviours
✓ sums probabilities to 1   ✓ forms quadratic equal to 0   ✓ solves quadratic, states unique value of k ✓ states probability

The random variable $W=5T-1$.

(b) Determine $E(W)$ and $\mathrm{Var}(W)$. (3 marks)

Solution
$E(W)=5E(T)-1=5\left(\frac{4}{5}\right)-1=3$ $\mathrm{Var}(T)=\frac{1}{5}\times\frac{4}{5}=\frac{4}{25}$ $\mathrm{Var}(W)=5^2\times\mathrm{Var}(T)=5^2\times\frac{4}{25}=4$
Specific behaviours
✓ $E(W)$ ✓ indicates $\mathrm{Var}(T)$ ✓ $\mathrm{Var}(W)$

Question 8 **(4 marks)**

The function f is defined by $f(x) = \dfrac{3}{x^2+27}$, so that $f''(x) = \dfrac{18(x^2-9)}{(x^2+27)^3}$.

(a) Describe the concavity of the graph of $y = f(x)$. (4 marks)

Solution
$f''(x) = 0 \Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3$ $x < -3, f''(x) > 0$ $-3 < x < 3, f''(x) < 0$ $x > 3, f''(x) > 0$ f is concave up when $x < -3$ and $x > 3$. f is concave down when $-3 < x < 3$.
Specific behaviours
✓ solves $f''(x) = 0$ ✓ indicates sign of $f''(x)$ in three intervals   ✓ states domains for concave up, down   ✓ uses correct inequalities in domains   *(penalise ambiguous language such as between -3 and 3, etc.)*

End of questions

Supplementary page

Question number: _________